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ABELIAN COMPLEX STRUCTURES AND SPECIAL GEOMETRIES

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Abstract

The construction of special geometric structures using solvable Lie groups is considered. Starting with a canonical structure on euclidean space, one translates it by using different Lie groups acting simply transitively. This procedure amounts to work directly at the Lie algebra level. We concentrate on the construction of special geometric structures related to a particular kind of invariant complex structures on Lie groups: the abelian ones. After an introduction with some survey results on abelian complex structures we give two applications, namely : weak hyperkähler with torsion structures on $\mathbb{R}^{4n}$ with underlying abelian hypercomplex structure and hypersymplectic structures on $\mathbb{R}^{4n}$ with underlying abelian complex product structure.

Key words: Complex, Structures complexstructures, Hyperkähler with torsion, Hypersymplectic.

Resumen

Estructuras abelianas coplejas y geometrías especiales. Se exhiben construcciones de estructuras geométricas especiales usando acciones de grupos de Lie solubles. Comenzando con una estructura canónica en el espacio euclídeo, se la traslada usando grupos de Lie diferentes que actúan simple y transitivamente en el mismo. Este procedimiento lleva a trabajar directamente en álgebras de Lie. Nos concentramos en la construcción de estructuras geométricas especiales relacionadas con una clase particular de estructuras complejas, las estructuras complejas abelianas. Luego de una introducción con resultados generales sobre estructuras complejas abelianas presentamos dos aplicaciones: estructuras hyperkähler con torsión en $\mathbb{R}^{4n}$ cuya estructura hipercompleja asociada es abeliana y estructuras hipersimplécticas en $\mathbb{R}^{4n}$ cuya estructura compleja producto asociada es abeliana.

Palabras clave: Estructuras complejas, Hyperkähler con torsión, Hipersimplécticas.

1. Introduction

In the past years various lines of research in mathematics have appeared related to special geometric structures that are shared by certain spaces which appear connected with physical problems. The picture provided by hamiltonian dynamics applies to many fields of physics due to its rich geometrical structure. Hamiltonian formalism is based on symplectic structures and a special and relevant class of symplectic manifolds are provided by Kähler manifolds where an extra compatible structure appears, that of a complex manifold. In relatively recent years, mathematicians on the one hand and theoretical or mathematical physicists on the other became interested in a special kind of Kähler structures, those called hyperkähler, due to the fact that are Kähler with respect to three

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complex structures which satisfy the law of quaternions. These structures are relevant in the description of (non abelian) monopoles [5], they bear a close connection with twistors and thus in particular to interesting classes of integrable systems [16]. More recently other special geometries like hyperkähler with torsion and hyper-symplectic (also called neutral-hyper-kähler) arose in a natural way in theoretical and mathematical physics. The hyperkähler with torsion structure is related to the existence of a metric connection having totally skew-symmetric torsion. For example, the geometry of such connections is present on the target space of supersymmetric sigma models with the Wess-Zumino term [26, 37, 38] and, in the supergravity theories, on the moduli space of a class of black holes [29]. Moreover, the geometry of NS-5 brane solutions of type II supergravity theories is generated by such a connection [43, 44, 42]. On the other hand hypersymplectic or neutral hyperkähler structures have significance in string theory. In [41], N=2 superstring theory is considered, showing that the critical dimension of such a string is 4 and that the bosonic part of the N=2 theory corresponds to self-dual metrics of signature (2,2) (see also [12] and [34]).

In all the previous special geometries, which do no not exhaust at all the many situations exhibiting a rich interplay between mathematics and physics, complex structures play a key role. Important results were obtained of known manifolds carrying such additional structures. Some examples are:

• Eguchi-Hanson [21] discovered a complete hyperkähler metric on the holomorphic cotangent bundle of CP^1. Its generalization to higher dimensions is due to Calabi [15].

• A coadjoint orbit of a complex Lie group G^c is a holomorphic symplectic manifold. These manifolds in many cases possess natural hyperkähler metrics due to the work of Kronheimer [36].

• A large class of homogeneous HKT and QKT manifolds, G/K, using an invariant metric on G and the canonical connection is exhibited in [40]. For this a decomposition of the Lie algebra of G is employed, which is most easily described in terms of colourings of Dynkin diagrams of simple Lie algebras. The colourings which give rise to HKT structures are found by using extended Dynkin diagrams.

• Compact complex surfaces with neutral hyperkähler metrics are biholomorphic to either complex tori or primary Kodaira surfaces and both carry non flat neutral hyperkähler metrics, by results of Kamada (see [35]). In higher dimensions, hypersymplectic structures on a class of compact quotients of 2-step nilpotent Lie groups were exhibited in [22] in their search of neutral Calabi-Yau metrics. Also in [3] hypersymplectic structures on $\mathbb{R}^{4n}$ with complete and not necessarily flat associated neutral metrics are exhibited.

In most cases the proof is based on the quotient construction. The quotient construction has its origins in the Marsden-Weinstein quotient construction in symplectic geometry but it was generalized to the Kähler and hyperkähler setting in [32], quaternionic and quaternionic Kähler setting in [27], hyperkähler with torsion setting in [28] among other situations.

One problem that often appears is to have an explicit description of the compatible metrics involved. To know about their completeness or their isometry groups is an important matter in this field.

In the past ten years approximately we have been working in the construction of certain kinds of special structures on manifolds but using actions of Lie groups (see for example [9], [17], [20]). We start with the canonical structure on euclidean space and then translate it by using different classes of groups acting simply transitively on it. This procedure amounts to work directly at the Lie algebra level. In many cases it is possible to exhibit the metric explicitly [3], [20] and to decide whether it is complete or not.

Throughout this note we will concentrate on the construction of special geometric structures related to a particular kind of invariant complex structures: the abelian ones. Abelian complex structures were introduced in [10] (see also [8] , [9]) in the context of Lie groups carrying many complex structures. A characterization of two-step nilpotent Lie algebras carrying abelian hypercomplex structures is given in [7] and a characterization of Lie algebras carrying abelian complex structures in terms of affine algebras appear in [11]. Abelian complex structures are in a sense complementary to complex strucures that make Lie groups into complex Lie groups. To make this last statement precise we recall that the differential of a (1,0)-form corresponding to an integrable complex structure has no (0,2)-component; if the complex structure makes G into a complex Lie group then the diferential of a (1,0)-form has only (2,0)-component and if the complex structure is abelian it has only (1,1)-component.

The organization of this paper is as follows. After a first section containing some general results on abelian structures, in the following sections, we give two applications, namely, weak hyperkähler with torsion structures on $\mathbb{R}^n$ with underlying abelian hypercomplex structure and hypersymplectic structures on $\mathbb{R}^n$ with underlying abelian product structure.

2. Abelian complex structures

A *complex structure* on a real Lie algebra $\mathfrak{g}$ is an endomorphism J of $\mathfrak{g}$ satisfying

$$J^2 = -Id, J[x,y] - [Jx,y] - [x,Jy] - J[Jx,Jy] = 0, \quad \forall x, y \in \mathfrak{g}. \qquad (1)$$

Note that complex Lie algebras are those for which the endomorphism J satisfies the stronger condition

$$J^2 = -Id, \quad J[x,y] = [x,Jy], \ \forall x, y \in \mathfrak{g} \qquad (2)$$

By a *hypercomplex structure* we mean a pair of anticommuting complex structures.

An *abelian* complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ satisfying

$$J^2 = -I, \quad [Jx,Jy] = [x,y], \quad \forall x, y \in \mathfrak{g} \qquad (3)$$

or equivalently

$$J^2 = -I, \quad [Jx,y] = -[x,Jy], \quad \forall x, y \in \mathfrak{g} \qquad (4)$$

By an abelian hypercomplex structure we mean a pair of anticommuting abelian complex structures. We observe that one can rewrite condition (1) as follows

$$J([x,y] - [Jx,Jy]) = [Jx,y] + [x,Jy] \quad \forall x, y \in \mathfrak{g}. \qquad (5)$$

Thus, it is easily obtained, using (3) and (4) that abelian complex structures are integrable. Note also that (5) implies that if a complex structure J satisfies $[x, y] - [Jx, Jy] \neq 0$ for some x, y then the commutator subalgebra has dimension ≥ 2. In particular,

Proposition 2.1. *If $\mathfrak{g}$ is a real Lie algebra with 1-dimensional commutator $[\mathfrak{g}, \mathfrak{g}]$ then every complex structure on $\mathfrak{g}$ is abelian (compare with Proposition 4.1 in* [9]*).*

Given a complex structure J on a Lie algebra $\mathfrak{g}$, the endomorphism J extends to the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, giving a splitting

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$$

where

$$\mathfrak{g}^{1,0} = \{X - iJX: X \in \mathfrak{g}\} \text{ and } \mathfrak{g}^{0,1} = \{X + iJX: X \in \mathfrak{g}\}$$

are complex Lie subalgebras of $\mathfrak{g}^{\mathbb{C}}$. Using (3) one verifies that abelian complex structures are those for which the subalgebras $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are abelian, and conversely.

There exist algebraic restrictions to the existence of abelian complex structures. In [19] it was proved that a real Lie algebra admitting an abelian complex structure must be solvable. Moreover, as a consequence of the next result, (for a proof see for example [4]).

Proposition 2.2. *If $\mathfrak{g}$ is a Lie algebra which admits a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ with $\mathfrak{g}_+$* and *$\mathfrak{g}_-$ abelian subalgebras, then $\mathfrak{g}$ is 2-step solvable* (i.e., *$\mathfrak{g}'$ is abelian*). One can obtain the following improvement.

Proposition 2.3. [45] *Let $\mathfrak{g}$ be a real Lie algebra admitting an abelian complex structure. Then $\mathfrak{g}$ is 2-step solvable.*

Proof. If $\mathfrak{g}$ is a real Lie algebra with an abelian complex structure then

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$$

is a sum of two abelian subalgebras, hence 2-step solvable. Since $(\mathfrak{g}^{\mathbb{C}})' = (\mathfrak{g})'^{\mathbb{C}}$ the proposition follows.

Example 2.4. The simplest examples of non abelian Lie algebras carrying abelian complex structures are provided by

i) aff($\mathbb{R}$), the Lie algebra of the affine motion group of $\mathbb{R}$(the bidimensional non-abelian Lie algebra), *aff*($\mathbb{R}$) = span$\{x, y\}$, with bracket $[x, y] = x$ and J given by $Jx = y$ and

ii) $\mathbb{R} \times h_n$, where h_n stands for the $2n+1$ – dimensional Heisenberg Lie algebra, $\mathbb{R} \times h_n = span\ \{w, z, x_i, y_i, i = 1, \dots, n\}$, with non zero bracket $[x_i, y_i] = z$ and J given by $Jz = w$, $Jx_i = y_i$, $i = 1, \dots, n$.

The Lie algebras introduced in *i)* and *ii)* have one dimensional commutator. Moreover, every Lie algebra with one dimensional commutator is a trivial central extension of one of these (see Theorem 4.1 in [9]). Hence we have obtained the following result:

Proposition 2.5. *Every even dimensional Lie algebra with one dimensional commutator carries an abelian complex structure.*

We have seen that Lie algebras carrying abelian complex structures need to be 2-step solvable with an even dimensional center. We show below examples of 2-step nilpotent algebras with even dimensional center which can not carry abelian complex structures.

Example 2.6. Two-step, free nilpotent Lie algebras are defined for each $r \geq 1$ as $f_r = V \oplus \Lambda^2 V$, V a 4r–dimensional real vector space and bracket given by $[x, y] = x \wedge y$ for $x, y \in V$. It is clear that $\Lambda^2 V$ is the center of the nilpotent Lie algebra. Assume an abelian complex structure J exists on f_r. Then, since abelian complex structures preserve the centre (in particular V and $\Lambda^2 V$ have to be even dimensional) one has a complex structure on $f_r / \Lambda^2 V$. Take W a complementary subspace to $\Lambda^2 V$ in f_r, invariant by J. Then $f_r = W \oplus \Lambda^2 W$ with J preserving the decomposition. In W take $\{x, Jx, y, Jy\}$ linearly independent, then $[x, y]$ and $[Jx, Jy]$ are linearly independent in the center, by the definition of the Lie algebra f_r. On the other hand $[x,y] = [Jx, Jy]$ since J is abelian. Thus no abelian complex structures exist on two-step, free nilpotent Lie algebras.

We next exhibit a complex structure on f_1 which can be easily generalized.

Let J be defined as follows

$Je_1 = e_2$, $Je_3 = e_4$, $J(e_1 \wedge e_2) = e_3 \wedge e_4$
$J(e_1 \wedge e_3) = e_2 \wedge e_3$, $J(e_1 \wedge e_4) = e_2 \wedge e_4$,

where $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathbb{R}^4$.

It can be easily checked that J defined as above is integrable in $f_1 = \mathbb{R}^4 \oplus \Lambda^2 \mathbb{R}^4$.

Generalizing the notion of complex and hypercomplex structures one has the following.

Definition 2.7. A Clifford structure of order m, or a Clifford C_m-structure, on a real Lie algebra $\mathfrak{g}$ is a family $\{J_\alpha\}_{\alpha=1,\dots,k}$ of endomorphisms of $\mathfrak{g}$ satisfying, if $1 \leq \alpha, \beta \leq m$,

1. $J_\alpha^2 = -I$, $J_\alpha J_\beta + J_\beta J_\alpha = 0$ $(\alpha \neq \beta)$
2. $N_\alpha(x,y) := [J_\alpha x, J_\alpha y] - J_\alpha [J_\alpha x, y] - J_\alpha [x, J_\alpha y] - [x,y] = 0$, $x, y \in \mathfrak{g}$, (integrability condition)
3. The subalgebra of End($\mathfrak{g}$) generated by $\{J_\alpha\}_{\alpha=1,\dots,k}$ has dimension 2^m.

Note that when $m = 1$ or $m = 2$, condition 1 in 2.7 automatically implies condition 3. This is no longer true when $m \geq 3$.

Remark. If one of the complex structures involved in the definition of Clifford structures is abelian then they are all abelian. The proof of this statement follows the lines of an analogous result proved in the hypercomplex case in [19].

The next construction, which appears in [10] shows that a class of two-step nilpotent Lie groups supports abelian Clifford structures.

Given $\mathfrak{g}$ a 2-step nilpotent Lie algebra, that is $[[\mathfrak{g},\mathfrak{g}]\mathfrak{g}] = 0$, set $d\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$. Define the following bracket on $d\mathfrak{g}$:

$$[x,y] = [(x_1, x_2), (y_1, y_2)] = ([x_1, y_1] + [x_2, y_2], 0)$$

Let J denote the endomorphism of $d\mathfrak{g}$ given by $J(x_1, x_2) = (-x_2, x_1)$. Clearly $J^2 = -I$.

Proposition 2.8. If $\mathfrak{g}$ is two-step nilpotent, then $d\mathfrak{g}$ is two-step nilpotent and J is integrable. More generally, for each $m \geq 1$, $d^m\mathfrak{g} = d(d^{m-1}\mathfrak{g})$ is a 2-step nilpotent Lie algebra carrying a C_m-structure.

Next, we define another kind of structure on a Lie algebra which is analogous to a complex structure. A *product structure* on $\mathfrak{g}$ is a linear endomorphism $E: \mathfrak{g} \to \mathfrak{g}$ satisfying $E^2 = 1$ (and not equal to ± 1 toghether with

$$E[X,Y] = [EX,Y] + [X,EY] - E[EX,EY] \quad \textit{for all } X,Y \in \mathfrak{g} \tag{6}$$

Given a product structure E on $\mathfrak{g}$, we have a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ into the direct sum of two linear subspaces, the eigenspaces associated to E.

The next definition relates complex and product structures.

Definition 2.9. *A complex product structure* on the Lie algebra $\mathfrak{g}$ is a pair $\{J, E\}$ of a complex structure J and a product structure E satisfying $JE = -EJ$.

The condition $JE = -EJ$ implies that the eigenspaces corresponding to the eigenvalues +1 and -1 of E have the same dimension.

The endomorphism $F = JE$ satisfies $F^2 = 1$, and overall $\{J,E,F\}$ obey the rules

$$-J^2 = E^2 = F^2 = 1, \qquad JE = F, \quad EF = -J, \quad FJ = E, \tag{7}$$

satisfied by the so-called paraquaternionic numbers. It is easy to verify that (6) is satisfied by F in place of E.

We refer to [6] for a thorough study of complex product structures on Lie algebras.

An almost product structure E on a Lie algebra $\mathfrak{g}$ satisfying $[Ex, Ey] = -[x,y]$ for all $x, y \in \mathfrak{g}$ will be called *abelian*. Related to these notions the following characterization appears in [3].

Proposition 2.10. Let $\{J,E\}$ be a complex product structure on the Lie algebra $\mathfrak{g}$ and let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be the associated double Lie algebra, i.e., $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are the Lie subalgebras of $\mathfrak{g}$ such that $E|_{\mathfrak{g}_+} = 1$, $E|_{\mathfrak{g}_-} = -1$. Then the following assertions are equivalent:

(i) J is an abelian complex structure.

(ii) The Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are abelian;

(iii) E is an abelian product structure.

If one of the conditions in the proposition above holds, we will say that the complex product structure $\{J,E\}$ is abelian.

3. Hyperkähler structures with torsion

On any hermitian manifold (M, J, g) there exists a unique connection ∇ satisfying $\nabla g = 0, \nabla J = 0$ and whose torsion tensor $c\ (X,Y,Z) = g\ (X, T(Y,Z))$ is totally skew-symmetric (i.e a three form). The torsion tensor of this connection is given by $c = -JdJF$, where $F = g(J.,.)$ is the Kähler form for J [23]. The geometry of such a connection is called by physicists a KT connection; among mathematicians this connection is known as the Bismut connection [14].

Let M be a smooth manifold with a hypercomplex structure $\{J_i\}_{i=1,2,3}$ and a riemannian metric g. M is said to be a hyperhermitian manifold if it is hermitian with respect to every J_i, $1 \leq i \leq 3$.

A given hyperhermitian manifold $(M,\{J_i\}_{i=1,2,3,}\ g)$ is an HKT (hyperkähler torsion) manifold ([37]) if there is a connection ∇ such that

$$\nabla g = 0, \qquad \nabla J_i = 0,\ i = 1,2,3,$$
$$c(X,\ Y,\ Z) = g(X,\ T(Y,Z)) \text{ is a three form.} \tag{8}$$

Such a connection is known as an HKT connection in physics literature; its geometry is known as an HKT geometry. HKT structures are called strong or weak depending on whether the torsion c is closed or not. Due to the uniqueness of the Bismut connection, a hyperhermitian manifold M will admit an HKT connection if and only if $J_1\ d\ J_1\ F_1 = J_2\ d\ J_2\ F_2 = J_3\ d\ J_3\ F_3$ (where F_i, $i = 1,2,3$ is the Kähler form associated to (J_i, g)) or equivalently if $\delta_{j1}(F_2 - i\ F_3) = 0$ [30]. If this connection exists, it is unique [23]. Moreover, by [39] the associated Lee forms $\theta_i = J_i d^* F_i$ coincide for $i = 1,2,3$.

Every 4-dimensional hyperhermitian manifold is HKT. If the dimension is 8, in [17] all simply connected nilpotent Lie groups which carry invariant abelian hypercomplex structures were obtained. There are three such groups and they are central extensions of Heisenberg type Lie groups. We show in [20] that abelian hypercomplex structures give rise to weak HKT structures on these groups (more generally on any Lie group) with respect to any compatible and invariant riemannian metric. These groups are diffeomorphic to $\mathbb{R}^8$. In coordinates $(x_1,\ldots,x_4, y_1,\ldots,y_4)$ the corresponding HKT metrics are given by:

$$\begin{aligned}
g_1 &= \Sigma dx_i^2 + \left(dy_1 - \frac{1}{2}(x_1dx_2 - x_2dx_1 - x_3dx_4 + x_4dx_3)\right)^2 + \\
&\quad \Sigma j \geq 2 dy_j^2,\\
g_2 &= \Sigma dx_i^2 + dy_i^2 + \left(dy_2 - \frac{1}{2}(x_1dx_3 - x_3dx_1 + x_2dx_4 + x_4dx_2)\right)^2 + \\
&\quad \left(dy_3 - \frac{1}{2}(x_1dx_4 - x_4dx_1 - x_2dx_3 + x_3dx_2)\right)^2 + dy_4^2,\\
g_3 &= \Sigma dx_i^2 + \left(dy_1 - \frac{1}{2}(x_1dx_2 - x_2dx_1 - x_3dx_4 + x_4dx_3)\right)^2 + \\
&\quad \left(dy_2 - \frac{1}{2}(x_1dx_3 - x_3dx_1 + x_2dx_4 - x_4dx_2)\right)^2 + \\
&\quad \left(dy_3 - \frac{1}{2}(x_1dx_4 - x_4dx_1 - x_3dx_2 - x_4dx_2)\right)^2 + dy_4^2.
\end{aligned}$$

These metrics have a transitive nilpotent group of isometries (hence they are complete) and they are non isometric to each other.

The 8-dimensional HKT structures obtained above are associated to abelian hypercomplex structures. One of the main results in [20] was to prove that on any 2-step nilpotent Lie groups all invariant HKT structures arise this way (see Theorem 3.1). Moreover the correspondence given in [7] between abelian hypercomplex structures on 2-step nilpotent Lie groups and subspaces of $sp(n)$, gives a method to construct infinitely many compact and non compact families of manifolds carrying non isometric HKT structures. By using this construction, in Section 4 of [20] it is shown that there exist non trivial deformations of homogeneous HKT structures on $\mathbb{R}^{4l}$, $l \geq 3$. Moreover, for rational parameters one obtains infinitely many HKT compact quotients of nilpotent Lie groups by discrete subgroups. This is in contrast with results in [13], [31] in the Kähler case.

4. Hypersymplectic structures

A hypersymplectic structure on a $4n$-dimensional manifold M is given by (J,E,g) where J, E are endomorphisms of the tangent bundle of M such that

$$J^2 = -1, \qquad E^2 = 1, \qquad JE = -EJ,$$

g is a neutral metric (that is, of signature (2n,2n)) satisfying

$$g(X,Y) = g(JX,\ JY) = -g(EX,\ EY)$$

for all X,Y vector fields on M, and the associated 2-forms

$$\omega_1(X,Y) = g(JX,Y), \quad \omega_2(X,Y) = g(EX,Y),$$
$$\omega_3(X,Y) = g(JEX,Y)$$

are closed. Manifolds carrying a hypersymplectic structure have a rich geometry, the neutral metric is Kähler and Ricci flat and its holonomy group is contained in $sp(2n,\mathbb{R})$ [33]. Moreover, the Levi Civita connection is flat, when restricted to the leaves of the canonical foliations associated to the product structure given by E [2]. Metrics associated to a hypersymplectic structure are also called neutral hyperkähler [35].

The quotient construction proved to be a powerful method to construct symplectic structures on manifolds. According to [33] this method cannot be always applied in the setting of hypersymplectic structures. In [3] we give a procedure to construct hypersymplectic structures on $\mathbb{R}^{4n}$ with complete and not necessarily flat associated neutral metrics. The most important feature achieved by this procedure is that the associated neutral metrics obtained will be complete and invariant by a 3-step nilpotent group of isometries (we note that homogeneity does not necessarily imply completeness in the pseudoriemannian setting.) The degree of nilpotency will be related to the flatness of the metric since we will show that the neutral metric is flat if and only if the group is at most 2-step nilpotent.

The idea behind the construction is to consider the canonical flat hypersymplectic structure on $\mathbb{R}^{4n}$ and then translate it by using an appropriate group acting simply and transitively on $\mathbb{R}^{4n}$. This group turns to be a double Lie group $(\mathbb{R}^{4n}, \mathbb{R}^{2n} \times \{0\}, \{0\} \times \mathbb{R}^{2n})$ constructed from affine data on $\mathbb{R}^{2n}$. This group naturally appears when one tries to obtain Lie algebras carrying abelian hypersymplectic structures. We will say that a hypersymplectic structure is abelian when the underlying complex product structure is abelian. Using Proposition 2.10 in Section 2 toghether with Theorems 3.4 and 3.5 in [1] one can show that any Lie algebra carrying an abelian hypersymplectic structure is a double product of two abelian Lie algebras endowed with compatible affine structures and symplectic forms.

Explicit examples of 3-step nilpotent Lie groups admitting compact quotients and carrying invariant complete and non flat hypersymplectic structures are also given in [3]. The induced metrics on the associated nilmanifold are neutral Kähler, complete, non-flat and Ricci flat.

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