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NONLOCAL DIFFUSION EQUATIONS

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Abstract

In this work we review some results concerning the asymptotic behaviour for nonlocal diffusion models of the form $u_t = J * u - u$ in the whole $\mathbb{R}^N$ or in a bounded smooth domain with Dirichlet or Neumann boundary conditions. In $\mathbb{R}^N$ we obtain that the long time behaviour of the solutions is determined by the behaviour of the Fourier transform J of near the origin, which is linked to the behaviour of J at infinity. If

$\hat{J}(\xi)=1-A|\xi|^{\alpha}+o(|\xi|^{\alpha})$ $(0<\alpha\leq 2)$, the asymptotic behaviour is the same as the one for solutions of the evolution given by the $\alpha/2$ fractional power of the Laplacian. In particular when the nonlocal diffusion is given by a compactly supported kernel the asymptotic behaviour is the same as the one for the heat equation, which is a local model. Concerning the Dirichlet problem for the nonlocal model we prove that the asymptotic behaviour is given by an exponential decay to zero at a rate given by the first eigenvalue of an associated eigenvalue problem with profile an eigenfunction of the first eigenvalue. Finally, we analyze the Neumann problem and find an exponential convergence to the mean value of the initial condition.

Keywords: Nonlocal diffusion, fractional Laplacian, Dirichlet boundary conditions, Neumann boundary conditions.

2000 Mathematics Subject Classification. 35B40, 45A05, 45M05.

Resumen

Ecuaciones de difusión no locales. En este trabajo haremos una revisión de algunos resultados recientes sobre el comportamiento asintótico de problemas de difusión no-locales de la forma $u_t = J * u - u$ en todo $\mathbb{R}^N$ o en un dominio acotado con condiciones de borde de tipo Dirichlet o Neumann. En $\mathbb{R}^N$ se obtiene que el comportamiento asintótico de las soluciones está determinado por el comportamiento de la transformada de Fourier de J cerca del origen, dicho comportamiento está relacionado con el de J cerca de infinito. Si $\hat{J}(\xi)=1-A|\xi|^{\alpha}+o(|\xi|^{\alpha})$ $(0<\alpha\leq 2)$, entonces el comportamiento asintótico es el mismo que el de las soluciones de la evolución que tiene como operador la potencia fraccionaria $\alpha/2$ del Laplaciano. En particular, cuando el operador no local tiene un núcleo de soporte compacto en comportamiento asintótico de las soluciones que es el mismo que el de las soluciones de la ecuación del calor, que es un modelo local. En cuanto al problema de Dirichlet para el operador no local se demuestra que se tiene un decaimiento exponencial a cero con una tasa dada por un primer autovalor y con un perfil dado por una autofunción asociada a este autovalor. Finalmente, también analizamos el problema de Neumann y demostramos la convergencia al valor medio de la condición inicial.

Palabras clave: Difusion no local, Laplaciano fraccionario, Condiciones de borde tipo Dirichlet, Condiciones de borde tipo Neumann.

2000 Mathematics Subject Classification. 35B40, 45A05, 45M05.

Introduction

The aim of this paper is to review some of the recent results obtained by the author contained in [8], see also [1], [2], [12], [13], [18] and [19].

We will study the asymptotic behaviour of solutions of a nonlocal diffusion operator in the whole $\mathbb{R}^N$ or in a bounded smooth domain with Dirichlet or Neumann boundary conditions.

First, let us introduce what kind of nonlocal diffusion problems we will consider. To this end, let $J:\mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial function with $\int_{\mathbb{R}^N} J(r)dr = 1$.

Nonlocal evolution equations of the form

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t),$$
$$u(x,0) = u_0(x), \quad (1.1)$$

and variations of it, have been recently widely used to model diffusion processes, see [3], [4], [5], [7], [10], [12], [16], [17], [20], [21] and [22]. As stated in [16], if $u(x,t)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then

$$(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$$

is the rate at which individuals are arriving to position x from all other places and

$$-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$$

is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1).

Equation (1.1), is called nonlocal diffusion equation since the diffusion of the density u at a point x and time t does not only depend on u(x,t), but on all the values of u in a neighborhood of x through the convolution term J*u. This equation shares many properties with the classical heat equation, $u_t = cu_{xx}$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed, [16]. However, there is no regularizing effect in general. For instance, if J is rapidly decaying (or compactly supported) the singularity of the source solution, that is a solution of (1.1) with initial condition a delta measure, $u_0 = \delta_0$, remains with an exponential decay. In fact, this fundamental solution can be decomposed as $w(x,t) = e^{-t}\delta_0 + v(x,t)$ where v(x, t) is smooth. In this way we see that there is no regularizing effect since the solution u of (1.1) can be written as $u = w * u_0 = e^{-t}u_0 + v * u_0$ with v smooth, which means that u(t) is as regular as u_0 is, and no more.

Let us also mention in passing that our results have a probabilistic counterpart in the setting of Markov chains.

Main results

Let us now state our results concerning the asymptotic behaviour for equation (1.1), for the Cauchy, Dirichlet and Neumann problems.

- *The Cauchy problem* - We will understand a solution of (1.1) as a function $u \in C^0\left([0,+\infty); L^1(\mathbb{R}^N)\right)$ that verifies (1.1) in the integral sense.

Our first result states that the decay rate as t goes to infinity of solutions of this nonlocal problem is determined by the behaviour of the Fourier transform of J near the origin. The asymptotic decays are the same as the ones that hold for solutions of the evolution problem with right hand side given by a power of the Laplacian.

In the sequel we denote by $\hat{f}$ the Fourier transform of f. Let us recall our hypothesis on the kernel J that we will assume throughout this paper,

$[(H)]\, J \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative, radial function with

$$\int_{\mathbb{R}^N} J(r)dr = 1$$

This means that J is a radial density probability which implies obviously that $|\hat{J}(\xi)| \le 1$ with $\hat{J}(0) = 1$, and we shall assume that $\hat{J}$ has an expansion of the form $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$ for $\xi \to 0\ (A > 0)$. Remark that in this case, (H) implies also that $0 < \alpha \le 2$ and $\alpha \neq 1$ if J has a first momentum.

Theorem I. *Let u be a solution of* (1.1) *with initial condition such that*

$$u_0, \hat{u}_0 \in L^1(\mathbb{R}^N)$$

If there exist A>0 and $0 < \alpha \le 2$ such that

$$\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \quad \xi \to 0,$$

then the asymptotic behaviour of u(x,t) is given by

$$\lim_{t\to+\infty} t^{N/\alpha} \max_x \left|u(x,t)-v(x,t)\right| = 0,$$

where v is the solution of

$$v_t(x,t) = -A(-\Delta)^{\alpha/2} v(x,t)$$

with initial condition v(x,0) = u_0(x). Moreover, we have

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\, t^{-N/\alpha},$$

and the asymptotic profile is given by

$$\lim_{t\to+\infty} \max_x \left|t^{N/\alpha} u(yt^{1/\alpha},t) - \|u_0\|_{L^1} GA(y)\right| = 0,$$

where G_A (y) satisfies

$$\hat{G}_A(\xi) = e^{-A|\xi|^\alpha}$$

In the special case $\alpha = 2$, the decay rate is t^{-N/2} and the asymptotic profile is a gaussian

$$G_A(y) = (4\pi A)^{N/2} \exp\left(-A|y|^2/4\right).$$

Note that in this case (that occurs, for example, when J is compactly supported) the asymptotic behaviour is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a gaussian.

The decay in L^∞ of the solutions together with the conservation of mass give the decay of the L^p-norms by interpolation. As a consequence of Theorem I, we find that this decay is analogous to the decay of the evolution given by the fractional Laplacian, that is,

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le C t^{-\frac{N}{\alpha}\left(1-\frac{1}{p}\right)}$$

we refer to [11] for the decay of the L^p -norms for the fractional laplacian, see also [14] and [15] for finer decay estimates of L^p -norms for solutions of the heat equation.

Next we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and impose boundary conditions to our model.

- *The Dirichlet problem* - We consider the problem

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), & x\in\Omega,\ t>0, \\ u(x,t) = 0 & x\notin\Omega,\ t>0, \\ u(x,0) = u_0(x), & x\in\Omega. \end{cases} \tag{1.3}$$

In this model we have that diffusion takes place in the whole $\mathbb{R}^N$ but we impose that u vanishes outside Ω. This is the analogous of what is called Dirichlet boundary conditions for the heat equation. However, the boundary data is not understood in the usual sense. As for the Cauchy problem we understand solutions in an integral sense.

In this case we find an exponential decay given by the first eigenvalue of an associated problem and the asymptotic behaviour of solutions is described by the unique (up to a constant) associated eigenfunction.

Let $\lambda_1 = \lambda_1(\Omega)$ be given by

$$\lambda_1 = \inf_{u\in L^2(\Omega)} \frac{\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(u(x)-u(y)\right)^2 dx\,dy}{\int_\Omega \left(u(x)\right)^2 dx}$$

and ϕ_1 an associated eigenfunction (a function where the infimum is attained).

Theorem II. *For every*

$$u_0 \in L^1(\Omega)$$

there exists a unique solution u of (1.3) such that

$$u \in C\left([0,\infty); L^1(\Omega)\right)$$

Moreover, if

$$u_0 \in L^2(\Omega)$$

solutions decay to zero as $t \to \infty$ with an exponential rate

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}.$$

If u_0 is continuous, positive and bounded then there exist positive constants C and C^* such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C e^{-\lambda_1 t}.$$

and

$$\lim_{t\to\infty} \max_x \left|e^{\lambda_1 t} u(x,t) - C^* \phi_1(x)\right| = 0.$$

- *The Neumann problem* - Let us turn our attention to Neumann boundary conditions. We study

$$\begin{cases} u_t(x,t) = \int_\Omega J(x-y)\left(u(y,t)-u(x,t)\right)dy, & x\in\Omega,\ t>0 \\ u(x,0) = u_0(x), & x\in\Omega. \end{cases} \tag{1.8}$$

Again solutions are to be understood in an integral sense. In this model we have that the integral terms take into account the diffusion inside Ω In fact, as we have explained the integral part of the equation takes into account the individuals arriving or leaving position x from other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The individuals may not enter nor leave the domain Ω. This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

Again in this case we find that the asymptotic behaviour is given by an exponential decay determined by an eigenvalue problem. Let β_1 be given by

$$\beta_1 = \inf_{u\in L^2(\Omega), \int_\Omega u=0} \frac{\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\left(u(y)-u(x)\right)^2 dx\,dy}{\int_\Omega \left(u(x)\right)^2 dx}$$

Concerning the asymptotic behaviour of solutions of (1.8) our last result reads as follows:

Theorem III. *For every*

$$u_0 \in L^1(\Omega)$$

there exists a unique solution u of (1.8) such that

$$u\in C\left([0,\infty); L^1(\Omega)\right)$$

This solution preserves the total mass in

Ω

$$\int_\Omega u(y,t)dy = \int_\Omega u_0(y)dy.$$

Moreover, let

$$\varphi = \frac{1}{|\Omega|}\int_\Omega u_0$$

be the mean value of the initial condition, then the asymptotic behaviour of solutions of (1.8) is described as follows: if

$$u_0 \in L^2(\Omega)$$

then

$$\left\|u(\cdot,t)-\varphi\right\|_{L^2(\Omega)} \le e^{-\beta_1 t}\left\|u_0-\varphi\right\|_{L^2(\Omega)},$$

and if u_0 is continuous and bounded there exist a positive constant C such that

$$\left\|u(\cdot,t)-\varphi\right\|_{L^\infty(\Omega)} \le Ce^{-\beta_1 t},$$

Comments

We will now devote some lines to comment on our results from the qualitative viewpoint, in order to give a clearer picture of the situation.

- Absence of regularization - As was said above, there is clearly NO regularizing effect, since the fundamental solution takes the form:

$$u(x,t) = e^{-t}\delta_0(x) + v(x,t).$$

The function v has no point singularity at x=0. Moreover, if

$$\hat{J} \in L^1\left(\mathbb{R}^N\right)$$

then

$$v \in C^\infty\left(\mathbb{R}^N \times \mathbb{R}_+\right)$$

This phenomenon is in sharp contrast with what happens for the heat equation, for which an initial condition like δ_0 is automatically regularized and the corresponding solution is C^∞.

One could think that this situation is in some sense close to what happens in the subcritical fast-diffusion case: $u_t = \Delta(u^m)$, with $0 < m \le (N-2)_+/N$. Indeed, it is proved in [6] that the solution with initial data $u_0 = \delta_0$ has a permanent singularity for all positive times, $u(x, t) = \delta_0(x) \otimes 1(t)$ which means that there is no diffusion at all for this special data.

But in fact, the nonlocal equation (1.1) is a little bit more interesting since some mass transfer occurs. Although the Dirac delta remains at x=0, its mass decays exponentially fast. Thus, total conservation of mass implies that this mass is redistributed in all the surrounding space, through the function v(x , t).

This may be seen as a radiation phenomena, which is a feature shared by the fast diffusion equation in the case $(N-2)_+/N < m < 1$. When considering strong singularities of the kind $\infty \cdot \delta_0$(see [9]), there is an explicit solution which reads

$$u(x,t) = \left(\frac{Ct}{|x|^2}\right)^{\frac{1}{1-m}}$$

Such a solution has also a standing singularity at x=0, but nevertheless radiation occurs. The only difference is that, in the fast diffusion situation, the singularity has an infinite mass, and the amount of mass spread into the surrounding space will eventually lead to $u(x, t) \to +\infty$ as $t \to \infty$ everywhere.

- Influence of the behaviour of J- Let us first notice that in the Cauchy problem, if J is compactly supported in $\mathbb{R}^N$, then it has a second momentum, $\int_{\mathbb{R}^N} |x|^2 J(x)dx < +\infty$, and since by symmetry the first momentum of J is null, we necessarily have

$$\hat{J}(\xi) = 1 - c|\xi|^2 + o\left(|\xi|^2\right), \qquad \xi \to 0,$$

which implies an asymptotic behaviour of heat equation type, which is quite surprising since the Heat Equation is a local equation.

The same happens even if J is not compactly supported, but decreases sufficiently fast at infinity (roughly speaking, faster than $|x|^{-(N+2)}$). A well-known example is provided by the Gaussian law, namely in 1-D,

$$J(x) = e^{-x^2}, \quad \hat{J}(\xi) = e^{-|\xi|^2} = 1 - |\xi|^2 + o\left(|\xi|^2\right), \quad \xi \to 0.$$

In general, J may not have a second momentum, so that more general expansions may occur:

$$\hat{J}(\xi) = 1 - c|\xi|^\alpha + o\left(|\xi|^\alpha\right)$$

with $\alpha \in (0, 2]$, like it is the case for stable laws of index α (see [D], p.149). A typical example (in 1-D) is the Cauchy Law,

$$J(x) = \frac{1}{1+|x|^2}, \text{ where } \hat{J}(\xi) = 1 - |\xi| + o(|\xi|), \xi \to 0.$$

Note that this example provides a J that does not have a first momentum but has nevertheless an expansion of the form $\hat{J}(\xi) = 1 - |\xi| + o(|\xi|)$. In these cases, we obtain that the asymptotic behaviour is given by the non-local fractional Laplace parabolic equation.

But more diffusions may be considered like for instance the case when

$$\hat{J}(\xi) \sim 1 - \xi^2 \ln \xi \quad \text{as} \quad \xi \to 0.$$

This last case is really interesting since it can be shown that the asymptotic behaviour is still given by a solution of the Heat Equation, yet viewed in a different time scale. More precisely, if $\hat{J}$ is as above and v is the solution of the Heat Equation $v_t = (1/2)\Delta v$ with the same initial datum, then

$$\lim_{t \to +\infty} (t \ln t)^{N/2} \max_x |u(x,t) - v(x, t \ln t)| = 0.$$

- On the diffusive effect of the equation - In the case when J has a moment of order 2, then $\hat{J}(\xi) = 1 - A|\xi|^2 + o\left(|\xi|^2\right)$ where A is defined as follows:

$$-\frac{1}{2} D^2 \hat{J}(0) = \left(\frac{1}{2N} \quad x^2 J(x) dx \right) \text{Id} = \text{A} \cdot \text{Id}.$$

Since the first moment of J is null, its second moment measures the dispersion of J around its mean, which is zero. Now, the asymptotic behaviour of solutions to (1.1) is related to those of the heat equation with speed c=A^{1/2}. This means that the more dispersed J is, the greater the speed.

This effect can be understood as follows: if J is not dispersed, then almost no diffusion occurs since J*u ~ u, the limit case being $J = \delta_0$ for which the equation becomes: $u_t = \delta_0 * u - u = 0$.

Thus for concentrated J's, the diffusion effect is very small, which is also visible in the asymptotic behaviour since the speed of the Gaussian profile is also small.

On the contrary, when J is very dispersed, (J*u)(x_0,t) will take into accounts values of the density u situated at points "far" from x_0 so that a great diffusion effect occurs. This is reflected in the asymptotic Gaussian profile which has a great velocity.

- The frequency viewpoint - A simple way to understand our results in the Cauchy problem is the following: the asymptotic behaviour that we have found means that at low frequencies ($\xi \sim 0$), the operator is very much like the fractional Laplacian (usual Laplacian if $\alpha = 2$). Now, as time evolves, diffusion occurs and high frequencies of the initial data go to zero, this si reflected in the explicit frequency solution:

$$\hat{u}(\xi, t) = e^{(\hat{J}(\xi)-1)t} \hat{u}_0(\xi).$$

Indeed, if J is a L^1 function, then it happens that $\hat{J}(\xi) \to 0$ as $\xi \to \infty$, so that for $|\xi| >> 1$, the high frequencies of u_0 are multiplied by something decreasing exponentially fast in time (this could be different in the case when J is a measure, but we do not consider such a case here).

Thus, roughly speaking, only low frequencies of the solution will play an important role in the asymptotic behaviour as $t \to \infty$, which explains why we obtain something similar to the fractional Laplacian equation (or heat equation) in the rescaled limit.

And in fact what it is done to prove Theorem I in [8] is precisely to separate the low frequencies where we use the expansion from the high frequencies that we control since they tend to zero fast enough in a suitable time scale.

- Asymptotics in bounded domains - In the case of bounded domains, the asymptotic behaviour of solutions is NOT related to the behaviour of $\hat{J}$ near zero. Indeed, this case is similar to the case when J is compactly supported, since the operator will not take into account values of u at $|x| = +\infty$. The asymptotic behaviour thus depends only on the eigenvalues of the operator (whether in Dirichlet or Neumann problems). However, if the domain is unbounded the behaviour of J at infinity may enter into play.

Possible extensions

Now we briefly comment on some possible extensions of our results.

1) First, concerning the Cauchy problem, one can study the behaviour of the solutions when the asymptotic expansion of $\hat{J}$ near the origin is not of the form $\hat{J}(\xi) = 1 - A|\xi|^{\alpha} + o(|\xi|^{\alpha})$

2) An interesting problem to look at is to study diffusions given by kernels that depend on x and y and not only on x-y. In this case our results do not apply since the use of the Fourier transform was the key of our arguments. Also, let us remark that our proofs strongly rely on hypothesis (H). It is interesting to known up to what extend (H) is necessary.

3) Another interesting problem is to look at the Dirichlet or Neumann problems in unbounded domains, for example in a half-space. In this case it is not clear what the asymptotic behaviour should be.

4) Finally, one may try to analyze discrete in space versions of these problems (like the ones considered in [4]) and see if they behave as their continuous counterpart. We believe that this is an interesting issue in order to develop numerical approximations for these problems.

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